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An Introduction to Generalized Linear Models

Solutions

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**CHAPTER 1**

***Exercise 1.1:***

**Let and be independent random variables with**  **and .   
If**  **and what is the joint distribution of and ?**

***SOLUTION:***

*A reminder from the book:*

1.4.1 Normal distributions:

1. If the random variable has the Normal distribution with mean and variance , its probability density function is:

We denote this by .

1. The Normal distribution with and , , is called the **standard Normal distribution**.
2. Let denote Normally distributed random variables with  for and let the covariance of and be denoted by:

where is the correlation coefficient for and . Then the joint distribution of the ’s is the **multivariate Normal distribution** with mean vector and variance-covariance matrix with diagonal elements and non-diagonal elements for . We write this as:

, where

1. Suppose the random variables are independent and normally distributed with the distributions for . If

where the ’s are constants. Then is also Normally distributed, so that:

It seems that the joint distribution of two normally distributed variables is yet another normal distri-bution. In this exercise, in order to find the joint distribution of and , we first need to determine the mean, the variance and the covariance of  and and then use those to derive the joint distribution.

Given that:

First, let us find the means of and :

Next, let us calculate the variances of and :

And finally, let us also compute the covariance between and :

Therefore, the joint distribution will be:

The correlation coefficient between and in this case shall be:

Therefore, another way to express the joint distribution, would be:

Where:

***Exercise 1.2:***

**Let and be independent random variables with and .   
a. What is the distribution of ?  
b.** **If , obtain an expression for . What is its distribution?  
c. If and its distribution is , obtain an expression for . What is its distribution?**

***SOLUTION:***

*A reminder from the book:*

1.4.2 Chi-squared distribution:

1. The **central chi-squared distribution** with degrees of freedom is defined as the sum of squares of independent random variables each with the standard Normal distri-bution. It is denoted by:

In matrix notation, if then so that .

1. If has the distribution , then its expected value is and its variance is .
2. If are independent Normally distributed random variables each with the distribu-tion then:

because each of the variables has the standard Normal distribution .

1. Let be independent random variables each with the distribution and let , where at least one of the ’s is non-zero. Then the distribution of:

has larger mean and larger variance than where . This is called the **non-central chi-squared distribution** with degrees of freedom and **non-centrality parameter** . It is denoted by .

1. Suppose that the ’s are not necessarily independent and the vector has the multivariate normal distribution where the variance-covariance matrix is non-singular and its inverse is . Then:
2. More generally if then the random variable has the non-central chi-squared distribution where .
3. If are independent random variables with the chi-squared distributions , which may or may not be central, then their sum also has a chi-squared distribution with degrees of freedom and non-centrality parameter , i.e.,

This is called the **reproductive property** of the chi-squared distribution.

1. Let , where has elements but the ’s are not independent so that is singular with rank and the inverse of is not uniquely defined. Let denote a generalized inverse of . Then the random variable has the non-central chi-squared distribution with degrees of freedom and non-centrality parameter .

**a.** As property 1 from above would suggest, the chi-squared distribution with degrees of freedom, , is the distribution of the sum of the squares of independent standard normal random variables. If  is a random variable following a normal distribution with mean and variance (), then the distribution of  is a special case of the **chi-squared distribution with one degree of freedom,** . Meaning that:

The chi-squared distribution with 1 degree of freedom is sometimes referred to as the exponential distribution with rate parameter (, ).

So, the distribution of is or equivalently, an exponential distribution with rate parameter .

**b.** The expression is the dot product of the vector with itself. So:

We know that and , and that they are independent. We also know (form a) that is a special case of the chi-squared distribution with one degree of freedom, , or in other words: .

Furthermore, we are given that: , thus:

Since both and are independent and follow a chi-squared distribution with 1 degree of freedom, then it follows that their sum will also follow the chi-squared distribution, but with two degrees of freedom, that are coming from the two terms combined. Therefore (and also according to property 3):

**c.** We know that and . Given that: and its distribution is , we have that:

The mean vector of , is:

While the Variance-Covariance matrix , is a diagonal matrix, because and are independent and it is:

Let us also compute the inverse of Variance-Covariance matrix , as it will be used:

Now, an expression for , will be:

As it was already shown above (in a), . Now, it was also shown (in b) that, and thus , plus a non-centrality parameter λ, which from property 6, is the following:

And therefore, since we are adding two chi-squared distributed variables, with one degree of freedom each, it follows that (again from property 6):

***Exercise 1.3:***

**Let the joint distribution of and be with:**

**a. Obtain an expression for . What is its distribution?**

**b.** **Obtain an expression for . What is its distribution?**

***SOLUTION:***

*A reminder from the book:*

1.4.2 Chi-squared distribution:

1. The **central chi-squared distribution** with degrees of freedom is defined as the sum of squares of independent random variables each with the standard Normal distri-bution. It is denoted by:

In matrix notation, if then so that .

1. If has the distribution , then its expected value is and its variance is .
2. If are independent Normally distributed random variables each with the distribu-tion then:

because each of the variables has the standard Normal distribution .

1. Let be independent random variables each with the distribution and let , where at least one of the ’s is non-zero. Then the distribution of:

has larger mean and larger variance than where . This is called the **non-central chi-squared distribution** with degrees of freedom and **non-centrality parameter** . It is denoted by .

1. Suppose that the ’s are not necessarily independent and the vector has the multivariate normal distribution where the variance-covariance matrix is non-singular and its inverse is . Then:
2. More generally if then the random variable has the non-central chi-squared distribution where .
3. If are independent random variables with the chi-squared distributions , which may or may not be central, then their sum also has a chi-squared distribution with degrees of freedom and non-centrality parameter , i.e.,

This is called the **reproductive property** of the chi-squared distribution.

1. Let , where has elements but the ’s are not independent so that is singular with rank and the inverse of is not uniquely defined. Let denote a generalized inverse of . Then the random variable has the non-central chi-squared distribution with degrees of freedom and non-centrality parameter .

**a.** First, let us compute the inverse of Variance-Covariance matrix , as it will be needed. So:

Since and , then their difference shall be:

And therefore, the joint distribution, will have the following form:

From property 5, we know that for a multivariate normal distribution , the quadratic form follows a chi-squared distribution with degrees of freedom equal to the dimension of y (and in this case, we have only two dimensions), and therefore:

**b.** From the previous question (a), we already know that:

And therefore, the expression for shall be:

Now, the distribution of is a more general case of the one described in the previous question (a) and thus follows property 6, meaning that: “if then the random variable has the non-central chi-squared distribution where .”

In our case, can be written as , where: . So if we expanded on this, we would have:

with:

* , because , and the quadratic form of a multivariate normal distribution follows a chi-squared distribution with degrees of freedom equal to the dimension of (which is 2).
* is normally distributed with a mean of 0.

Thus is a sum of a chi-squared distribution and a normal distribution. This means that follows a non-central chi-squared distribution with 2 degrees of freedom and a non-centrality parameter , which is:

Therefore, in conclusion:

***Exercise 1.4:***

**Let be independent random variables each with the distribution . Let:**

**a. What is the distribution of ?**

**b. Show that .**

**c. From (b) it follows that . How does this allow you to deduce that and are independent?**

**d. What is the distribution of ?**

**e. What is the distribution of ?**

***SOLUTION:***

**a.** Since the​ are independent and each has the distribution , the expectation of is:

while its variance is:

We know that consists of a linear combination of independent, normally distributed variables and therefore it is itself normally distributed. Thus, the distribution of shall be:

**b.** Let us start from the definition of the sample variance:

**c.** We